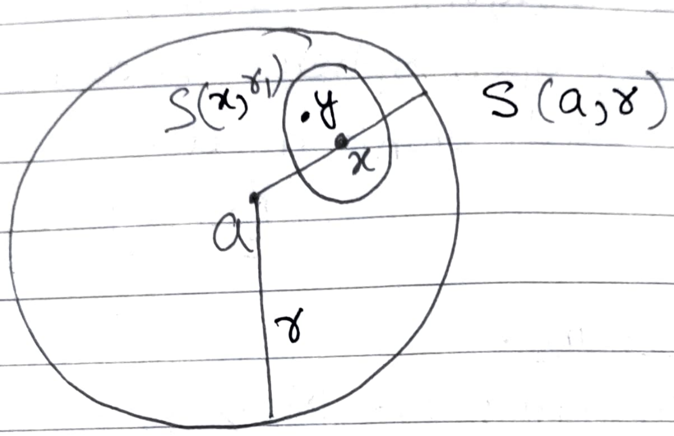
Neighbourhood of a point in Metric Space

Let (X, d) be a metric space and $a \in X$. A subset N_a of 'X' is called neighbourhood of point $a \in X$ if there exists an open sphere $S(a, r)$ centred at 'a' and radius 'r' such that $a \in S(a,r) \subset N_a$ for some $r > 0$.

Theorem : Every open sphere is a neighbourhood of each of its points.

Proof :



Let $S(a,r)$ be an open sphere and let $x \in S(a,r)$ be arbitrary. If $x = a$, then $a \in N_a \subseteq S(a,r)$

Therefore suppose that $x \neq a$.

In order to ~~shat~~ show that $S(a,r)$ is a neighbourhood of 'x', we must show that there exists $r_1 > 0$ such that

$$S(x, r_1) \subseteq S(a, r)$$

$\because$ $x \in S(a,r)$, therefore $d(x,a) < r$.

then $r_1 = r - d(x,a) > 0$

Let $y \in S(x, r_1)$ so that $d(y, x) < r_1$

Now, by using triangle inequality.

$$d(y,a) \leq d(y,x) + d(x,a)$$
$$< r_1 + d(x,a) = r_1 + (r - r_1)$$
$$= r$$

$\Rightarrow d(y,a) < r$

$\Rightarrow y \in S(a,r)$

Therefore $y \in S(x, r_1)$

$\Rightarrow y \in S(a,r)$

Hence $S(x, r_1) \subseteq S(a, r)$

$\Rightarrow$ Every open sphere is a neighbourhood of each point in it.

Proved

Theorem :

Statement : If M is a neighbourhood of α and $M \subset N$, then 'N' is also a neighbourhood of α.

Proof : Since 'M' is a neighbourhood of α, therefore for som $\varepsilon > 0$, we must have $S(\alpha, \varepsilon) \subset M$.. But it is given $M \subset N$.

$\therefore$ We have $S(\alpha, \varepsilon) \subset N$.

$\Rightarrow$ N is also neighbourhood of N

Proved

Theorem

Statement: If 'M and N are neighbourhoods of a point 'α' then $M \cap N$ is also a neighbourhood of 'α'.

Proof: Since 'M' and 'N' are neighbourhoods of 'α'. Therefore by definition we have, $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that

$$S(\alpha, \varepsilon_1) \subset M \text{ and } S(\alpha, \varepsilon_2) \subset N$$

Let $\varepsilon = \min.(\varepsilon_1, \varepsilon_2)$

then $S(\alpha, \varepsilon) \subset S(\alpha, \varepsilon_1) \subset M$ — (1)

and $S(\alpha, \varepsilon) \subset S(\alpha, \varepsilon_2) \subset N$ — (2)

$\therefore$ (1) & (2) implies

$$S(\alpha, \varepsilon) \subset M \cap N$$

i.e., $M \cap N$ is neighbourhood of 'α'.

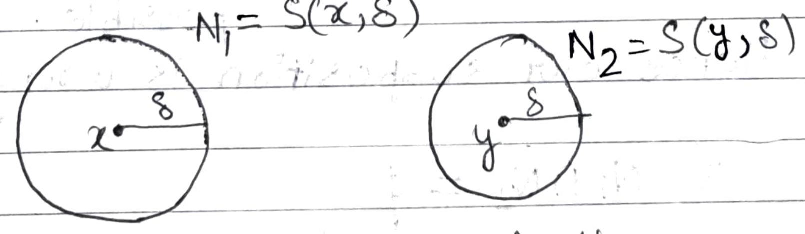
Proved

Theorem

Statement: Let (X, d) be a metric space. Then for each pair of distinct points 'x' & 'y' of 'X', there exists neighbourhoods N_1 & N_2 of 'x' & 'y' respectively such that

$$N_1 \cap N_2 = \phi.$$

Proof:



Here, we need to prove that there exists two open spheres say $S(x, \delta)$ & $S(y, \delta)$ with centres x & y respectively such that

$$S(x, \delta) \cap S(y, \delta) = \phi$$

Let (X,d) be a metric space.
then $x, y \in X$ and $x \neq y$.
$\Rightarrow d(x,y) > 0$ —— by [M1]
let $d(x,y) = r$.
then $\frac{r}{3} > 0$. Take $\delta = r/3$

Now, since N_1 & N_2 are neighbourhoods of 'x' & 'y' respectively we take them as open spheres
thus $N_1 = S(x,\delta)$ and $N_2 = S(y,\delta)$.

Let us suppose if possible $N_1 \cap N_2 \neq \phi$ and $p \in N_1 \cap N_2$.
$\because p \in N_1 \cap N_2$
$\Rightarrow p \in N_1$, $d(p,x) < \delta$
Also $p \in N_2$, $d(p,y) < \delta$.
then by triangle inequality we have.

$$d(x,y) \leq d(x,p) + d(p,y)$$
$$< \delta + \delta = 2\delta = \frac{2}{3} r$$

$\Rightarrow d(x,y) < \frac{2}{3} r$.

$\Rightarrow d(x,y) < \frac{2}{3} d(x,y)$ which is not possible.

Hence our supposition is wrong.

$\Rightarrow N_1 \cap N_2 = \phi$

Proved